The quantum equivalent of the gyromagnetic ratio for a spin-1/2 particle is given by $\gamma = gq/(2m)$.

Questions: a) At time $t = 0$, the observable S_x is measured with the result $+\hbar/2$. What is the state vector $|\psi\rangle$ immediately after the measurement?

When the measurement is made the state collapses onto the ψ_{x+} state. In the normalized z-representation this can be written

$$\boxed{|\psi(0)\rangle = \psi_{x+} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}}$$

b) Immediately after that S_x measurement, magnetic field $\mathbf{B} = B_0\hat{z}$ is applied, with the quantum state of the particle henceforth evolving in time until a time T. What is the state of the system at $t = T$?

We start by finding the Hamiltonian:

$$H = -\mu \cdot \mathbf{B} = \gamma \mathbf{B} \cdot \mathbf{S} = \frac{gq}{2m} B_0 S_z = \frac{gq}{2m} B_0 \frac{\hbar}{2}\sigma_z = -\frac{gq\hbar B_0}{4m}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

The time evolution operator is given by

$$e^{-iHt/\hbar} = \begin{bmatrix} e^{i\frac{gqB_0}{4m}t} & 0 \\ 0 & e^{-i\frac{gqB_0}{4m}t}\end{bmatrix}$$

We can simplify this by rewriting in terms of the Larmor frequency

$$\omega_0 \equiv \gamma B_0 = \frac{gqB_0}{2m}$$

to get

$$e^{-iHt/\hbar} = \begin{bmatrix} e^{\frac{i\omega_0 t}{2}} & 0 \\ 0 & e^{\frac{-i\omega_0 t}{2}}\end{bmatrix}$$

When this operates on the initial state $|\psi(0)\rangle$ we get

$$
\begin{aligned}
|\psi(T)\rangle &= e^{-iHT/\hbar}\,|\psi(0)\rangle \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} e^{\frac{i\omega_0 T}{2}} & 0 \\ 0 & e^{-\frac{i\omega_0 T}{2}} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} e^{\frac{i\omega_0 T}{2}} \\ e^{-\frac{i\omega_0 T}{2}} \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\left[e^{\frac{i\omega_0 T}{2}}\,|\uparrow\rangle + e^{-\frac{i\omega_0 T}{2}}\,|\downarrow\rangle\right]
\end{aligned}
$$

at some specific time $t = T$.

c) At time $t = T$, the magnetic field is very quickly changed to $\mathbf{B} = B_0\hat{y}$. After another time interval T, a measurement of S_x is carried out once more. What is the probability that a value $+\hbar/2$ is found?

The new Hamiltonian is

$$
H^\star = -\hat{\mu}\cdot\mathbf{B} = -\frac{gq\hbar B_0}{4m}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -\frac{\omega_0\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}
$$

So we must calculate

$$
|\psi(2T)\rangle = e^{-iH^\star T/\hbar}\,|\psi(T)\rangle = e^{\frac{i\omega_0 T}{2}}\,|\psi(T)\rangle
$$

In order to exponentiate the Hamiltonian we rewrite the Pauli matrix using the diagonalization procedure:

$$
e^{\sigma} = Pe^{D}P^{-1}
$$

where

$$
D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
$$

is the diagonal matrix whose entries are the eigenvalues of σ_y and

$$
P = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}
$$

are the matrices whose columns comprise the corresponding eigenvectors and its inverse, respectively. Therefore

$$
\begin{aligned}
|\psi(2T)\rangle &= e^{-iH^\star T}\,|\psi(T)\rangle \\
&= \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{\frac{i\omega_0 T}{2}} & 0 \\ 0 & e^{-\frac{i\omega_0 T}{2}} \end{bmatrix} \frac{1}{2} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} e^{\frac{i\omega_0 T}{2}} \\ e^{-\frac{i\omega_0 T}{2}} \end{bmatrix}
\end{aligned}
$$

which after some trigonometry and a fair amount of simplification (re-normalization is not necessary since $e^{iH^\star T}$ is unitary) comes to

$$
= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + i\sin(\omega_0 T) \\ \cos(\omega T) \end{bmatrix}
$$

We want the probability that the particle is in the $s_x = \frac{\hbar}{2}$ state, so we need to calculate

$$
\begin{aligned}
\langle P_{x+}\rangle &= \langle \psi(2T)|P_{x+}|\psi(2T)\rangle \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 - i\sin(\omega_0 T) & \cos(\omega_0 T) \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + i\sin(\omega_0 T) \\ \cos(\omega T) \end{bmatrix}
\end{aligned}
$$

where

$$
P_{x+} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
$$

is the projection matrix for the ψ_{x+} state expressed in the z-basis. This gives a final result of

$$
\langle P_{x+}\rangle = \frac{1}{2} + \frac{1}{2}\cos^2(\omega_0 T) = \boxed{\frac{3}{4} + \frac{1}{4}\cos(2\omega_0 T)}
$$

So the probability oscillates between 0.5 and 1 with a frequency equal to twice the Larmor frequency.